

L4.1 1, 2, ..., n qubits

● Pure State

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$$

$$\langle\psi|\psi\rangle = 1, \text{ global phase}$$

$$4 \in \mathbb{C} \Rightarrow 8 \in \mathbb{R} - 2$$

6 parameters

⚠ more than 2 Bloch spheres

$$n\text{-qubits } 2 \cdot 2^n - 2 = 2^{n+1} - 2$$

● Mixed States

$$\rho = |\psi\rangle\langle\psi|$$

$$\rho = \begin{pmatrix} \rho_{00} & \dots & \rho_{03} \\ \vdots & & \vdots \\ \rho_{30} & & \rho_{33} \end{pmatrix}$$

$$\rho^\dagger = \rho \quad \text{tr}(\rho) = 1$$

n-qubits $2^n \cdot 2^n - 1 = 2^{2n} - 1$

L4.2

$$s_{x_1}^1 = \text{tr}(\beta \sigma_{x_1}^1), s_y^1 = \dots, s_z^1$$

$$s_{x_1}^2, s_y^2, s_z^2$$

$$s_{xx} = \text{tr}(\beta \sigma_{x_1}^1 \sigma_{x_1}^2),$$

$$s_{xy} = \dots$$

$$s_{xz} = \dots$$

$$s_{yx} = \dots$$

⋮

	pure	mixed
1	2	3
2	6	15
3	14	63
4	30	255
n	$2^{n+1} - 2$	$2^{2n} - 1$

L4.3

Quantum Gates

Unitary Evolution

 U $2^n \times 2^n$ matrix

$$U^\dagger U = \mathbb{1} \quad U^\dagger = U^{-1}$$

$$U = e^{-iHt/\hbar}$$

single qubit gates

$$\boxed{X} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\boxed{Y} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\boxed{Z} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boxed{H} \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\boxed{S} \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

arbitrary angle rotations

$$\begin{aligned}
 R_x(\theta) &= e^{-i\theta\sigma_x/2} = \cos\left(\frac{\theta}{2}\right)\mathbb{1} - i\sin\left(\frac{\theta}{2}\right)\sigma_x \\
 &= \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -i\sin\left(\frac{\theta}{2}\right) \\ -i\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 R_y(\theta) &= e^{-i\theta\sigma_y/2} = \cos\left(\frac{\theta}{2}\right)\mathbb{1} - i\sin\left(\frac{\theta}{2}\right)\sigma_y \\
 &= \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}
 \end{aligned}$$

$$\begin{aligned}
 R_z(\theta) &= e^{-i\theta\sigma_z/2} = \cos\left(\frac{\theta}{2}\right)\mathbb{1} - i\sin\left(\frac{\theta}{2}\right)\sigma_z \\
 &= \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}
 \end{aligned}$$

phase gate S is a rotation by $\frac{\pi}{2}$
 around the z -axis and
 a phase $e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$

L4.5 theorem

if U is an arbitrary single qubit operation, i.e. unitary 2×2 matrix

there exists an angle θ , an axis \hat{n} and a phase $e^{i\alpha}$, such that

$$U = e^{i\alpha} R_{\hat{n}}(\theta)$$

with $\alpha, \theta \in \mathbb{R}$ and $n_x^2 + n_y^2 + n_z^2 = 1$
 $n_x, n_y, n_z \in \mathbb{R}$

in words; every single qubit operation is a rotation

$$R_{\hat{n}}(\theta) = e^{-i\theta \hat{n} \cdot \vec{\sigma} / 2}$$

$$= \cos\left(\frac{\theta}{2}\right) \mathbb{1} - i \sin\left(\frac{\theta}{2}\right) (n_x \sigma_x + n_y \sigma_y + n_z \sigma_z)$$

Example: Hadamard

L 4.6

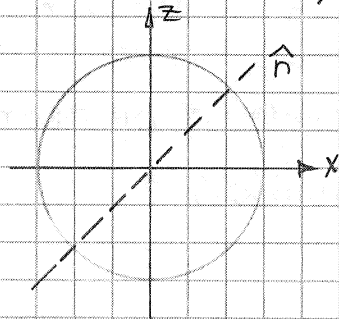
$$|0\rangle \xrightarrow{H} \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

$$|1\rangle \xrightarrow{H} \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$$

H changes from z to x basis

$$H^2 = \mathbb{1}$$

$$H = e^{i\frac{\pi}{2}} R_{\hat{n}}(\pi) \quad \hat{n} = \frac{1}{\sqrt{2}} (1, 0, 1)$$



X-Y decomposition

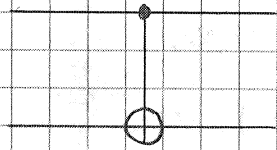
$$U = e^{i\alpha} R_x(\beta) R_y(\gamma) R_x(\delta)$$

every qubit operation can be performed with rotations around the x and y axes

L4.7

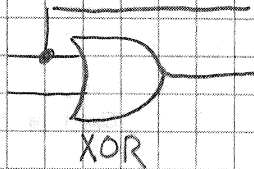
two qubit gates

controlled not

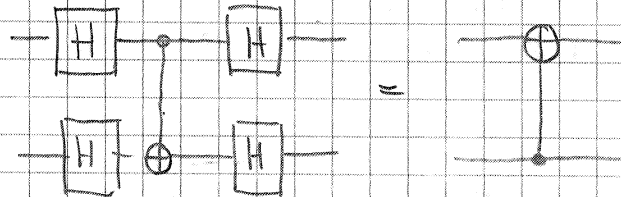


$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

classical equivalent



warning: do not think "classical"

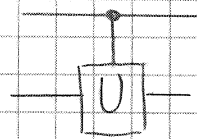


wrongly one could think that
the first qubit just is affected
by two Hadamard gates, i.e. the
identity

theorem

single qubit rotations and CNOT
are universal, i.e. every unitary
operation can efficiently be
approximated with $R_n(\theta)$ and CNOT

other two qubit gates



$$\text{controlled } U = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & U_{00} & U_{01} \\ & & U_{10} & U_{11} \end{pmatrix}$$

$$\text{SWAP} = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}$$

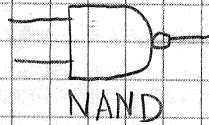
$$\sqrt{\text{SWAP}} = \begin{pmatrix} 1 & & & \\ & \frac{1+i}{2} & \frac{1-i}{2} & \\ & \frac{1-i}{2} & \frac{1+i}{2} & \\ & & & 1 \end{pmatrix} \quad \sqrt{\text{SWAP}} \text{ is universal}$$

$\sqrt{\text{SWAP}}$ is universal

L4.9

Comparison with Classical Logic

Boolean Logic



0	0	1
0	1	1
1	0	1
1	1	0

NAND is universal for Boolean logic
(single gate is enough)

note: for quantum gates
the x-axis is always the
time axes

L4.10

Deutsch Algorithm

find out if a coin is fair or fake

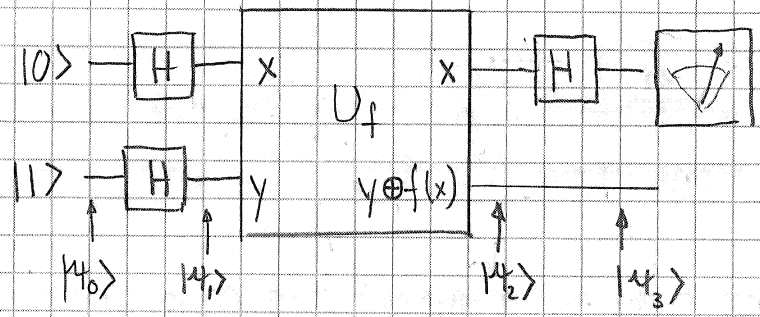
find out if a function is constant
or balanced (equal times 0 and 1)

$$f: \{\text{front, back}\} \rightarrow \{\text{head, tail}\}$$

$$f: \{0, 1\} \rightarrow \{0, 1\}$$

$f(0)$	$f(1)$		
0	0	const	$f(0) = f(1)$
0	1	$f(x) = x$, balanced	$f(0) \neq f(1)$
1	0	$f(x) = 1 - x$, balanced	$f(0) \neq f(1)$
1	1	const	$f(0) = f(1)$

(see Nielsen, Chuang pp 28-34)



$$U_f : |x, y\rangle \longrightarrow |x, y \oplus f(x)\rangle$$

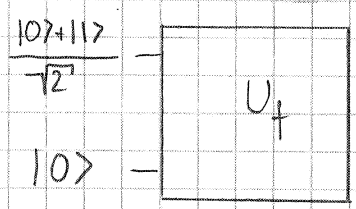
\oplus is the addition modulo 2

$$U_{f(x)=0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad U_{f(x)=1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$U_{f(x)=x} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad U_{f(x)=1-x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

CNOT

quantum parallelism



$\frac{|0, f(0)\rangle + |1, f(1)\rangle}{\sqrt{2}}$
 super position of results

L4.12

$$|\psi_0\rangle = |0\rangle|1\rangle$$

$$|\psi_1\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

constant
 $f(0)=f(1)$

balanced
 $f(0) \neq f(1)$

$$|\psi_2\rangle = \pm \frac{|0\rangle+|1\rangle}{\sqrt{2}} \frac{|0\rangle-|1\rangle}{\sqrt{2}} \quad \left| \psi_2 \right\rangle = \pm \frac{|0\rangle-|1\rangle}{\sqrt{2}} \frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

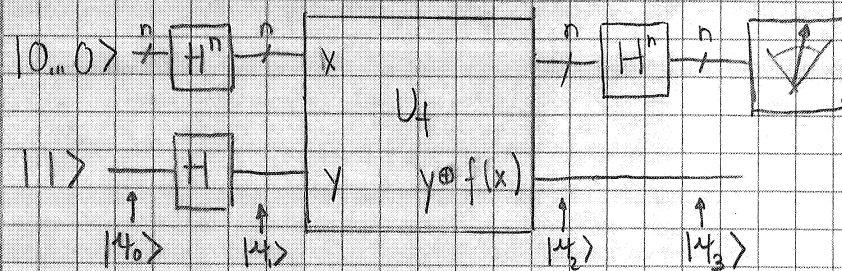
note

$$|x\rangle \frac{|0\rangle-|1\rangle}{\sqrt{2}} \xrightarrow{U_f} (-1)^{f(x)} |x\rangle \frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

$$|\psi_3\rangle = \pm |0\rangle \frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

$$|\psi_3\rangle = \pm |1\rangle \frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

L4.13 Deutsch Jozsa



$$f: \{0 \dots 00, 0 \dots 01, \dots, 1 \dots 11\} \rightarrow \{0, 1\}$$

$$H^n = H \otimes H \otimes \dots \otimes H$$

example $H^2 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

$$|u_1\rangle = \frac{|0 \dots 00\rangle + |0 \dots 01\rangle + \dots + |1 \dots 11\rangle}{\sqrt{2^n}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$= \sum_{x \in \{0,1\}^n} \frac{|x\rangle}{\sqrt{2^n}} \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|u_2\rangle = \sum_{x \in \{0,1\}^n} \frac{(-1)^{f(x)} |x\rangle}{\sqrt{2^n}} \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

(see Nielsen Chuang pp 34-36)

L5.1

Grover algorithm (Nielsen Chuang, chapter 6)

search in unsorted database

e.g search for number - telephonebook

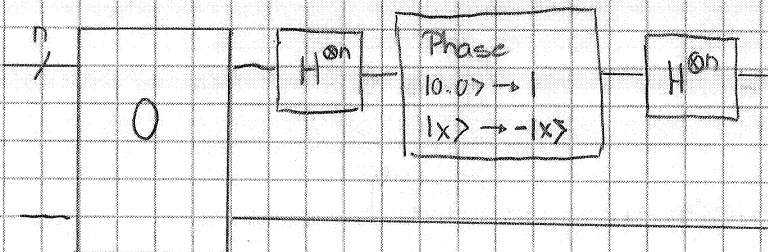
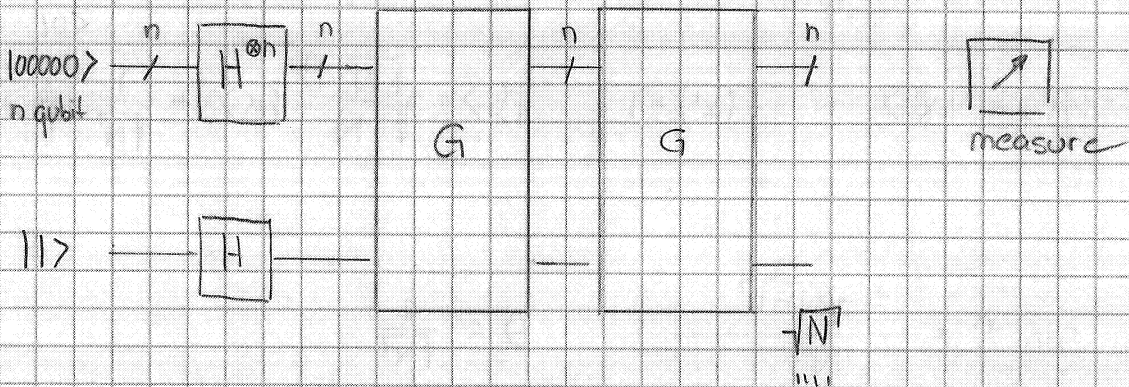
$$f: \{\text{index}\} \rightarrow \{\text{not-found}, \text{found}\}$$

$$f: \{0\dots 00, 0\dots 01, \dots, 11\dots 1\} \rightarrow \{0, 1\}$$

$$N = 2^n$$

$$|x\rangle |y\rangle \xrightarrow{0} |x\rangle |y \oplus f(x)\rangle$$

$$|x\rangle \xrightarrow{\frac{|0\rangle - |1\rangle}{\sqrt{2}}} \xrightarrow{0} (-1)^{f(x)} |x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$



$$Ph: \begin{bmatrix} 1 & & \\ & -1 & \\ & & \ddots \end{bmatrix} = 2 |0\rangle\langle 0| - \mathbb{1}$$

$$H^{\otimes n} (2 |0\rangle\langle 0| - \mathbb{1}) H^{\otimes n} = 2 |\xi\rangle\langle \xi| - \mathbb{1}$$

$$|\xi\rangle = H^{\otimes n} |0\rangle = \frac{1}{\sqrt{2^n}} \sum |x\rangle = \frac{1}{\sqrt{2^n}} (|0\dots 0\rangle + |0\dots 01\rangle + \dots + |1\dots 1\rangle)$$

$$2|\xi\rangle\langle\xi| - \mathbb{1}$$

$$\sum_k \alpha_k |k\rangle \rightarrow \sum_k (2\langle\alpha\rangle - \alpha_k) |k\rangle \quad \text{where}$$

$$\langle\alpha\rangle = \frac{1}{N} \sum_k \alpha_k$$

rotation about the mean

$$G = (2|\xi\rangle\langle\xi| - \mathbb{1})^O$$

geometric visualization

$$|\alpha\rangle = \frac{1}{\sqrt{N-1}} \sum_{\substack{x \\ f(x)=0}} |x\rangle$$

$$|\beta\rangle = |x'\rangle \quad f(x')=1 \quad |\xi\rangle = \sqrt{\frac{N-1}{N}} |\alpha\rangle + \frac{1}{\sqrt{N}} |\beta\rangle$$

$$\sin \theta = \frac{2\sqrt{N-1}}{N} \quad \theta \approx \frac{2}{\sqrt{N}}$$

\sqrt{N} iterations

quadratic speedup $O(\sqrt{N})$

probabilistic

optimal

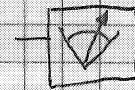
L4.14

Di Vincenzo Criteria

- Well defined qubits, scalable
- Initialization to a pure state
 $|00\dots 0\rangle$
- Universal set of quantum gates



- Qubit-specific measurement with high efficiency



- Long coherence time